Root loci

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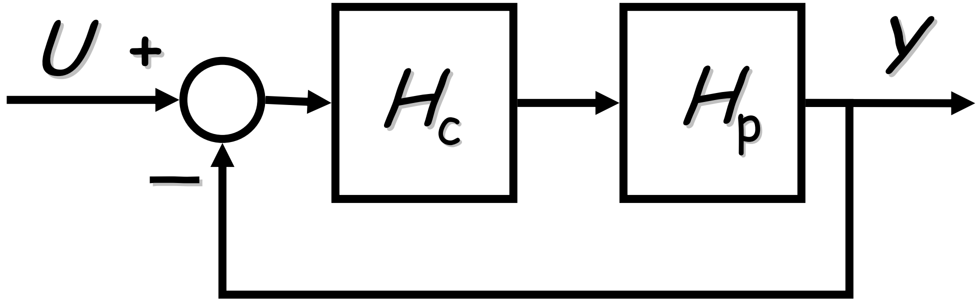
Open and closed systems



$$H_p = \frac{1}{s+1}$$

$$H_c = K$$

Loop transfer function:

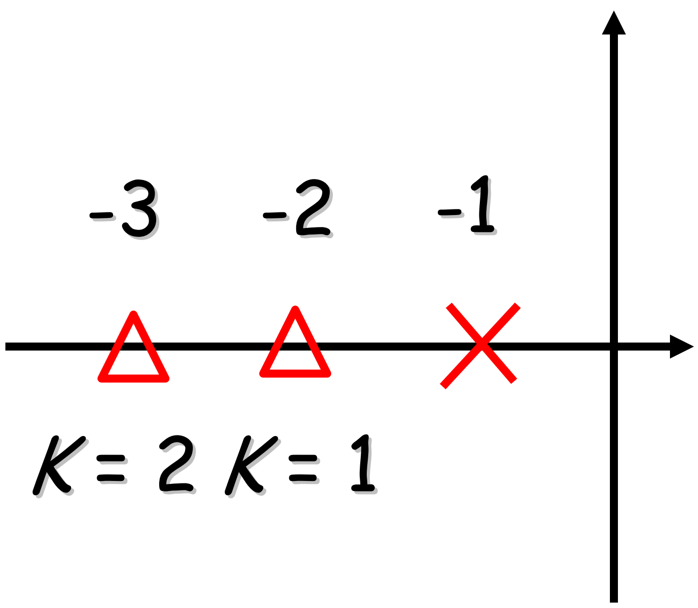
$$H_L = -H_c H_p = -K_L G_L = -K_L \frac{1}{s+1}$$

Y/U (closed loop): $$H = \frac{H_c H_p}{1 - H_L} = \frac{K_L}{s + (1 + K_L)}$$

Demo

$$H_{closed} = \frac{K}{s + (1 + K)}$$

gain: $\dfrac{K}{1+K}$



pole in: $s = -(1+K)$

20-sim

demo

First order system

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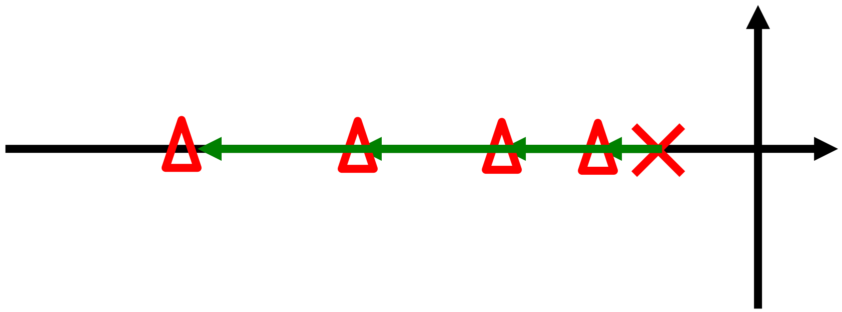
In a feedback system:

- when *K* increases, pole goes further to the left
- response is faster
- accuracy is better



Demo (second order)

$$H_p = \frac{K'}{(s+1)(s+3)} \longrightarrow$$

$$H = \frac{\dfrac{K'}{(s+1)(s+3)}}{1+\dfrac{K'}{(s+1)(s+3)}}$$

$$H = \frac{K'}{(s+1)(s+3)+K'}$$

$$H = \frac{K'}{s^2+4s+3+K'}$$

closed-loop poles:

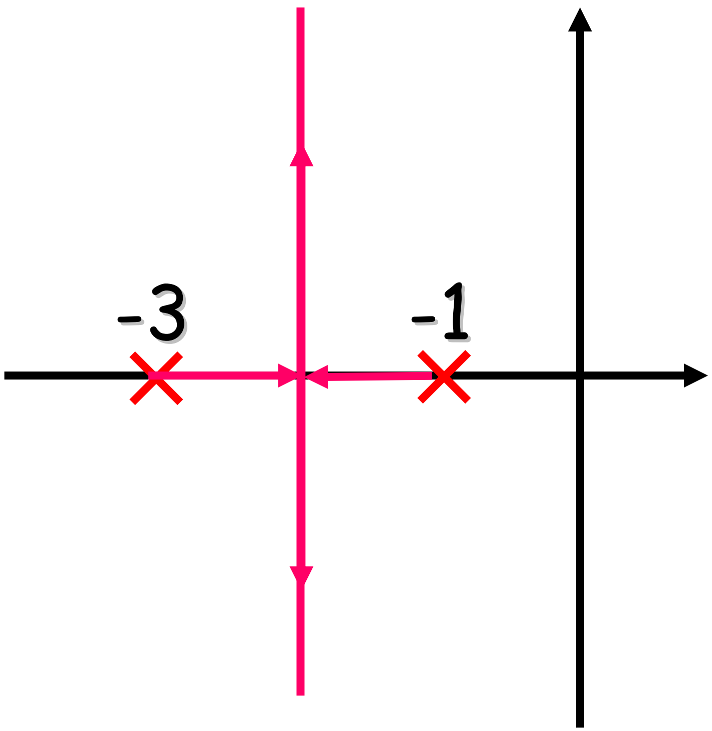
$$p_{1,2} = \frac{-4 \pm \sqrt{16-4(3+K')}}{2}$$

$$p_{1,2} = -2 \pm \sqrt{1-K'}$$

20-sim

Demo (second order)

$$p_{1,2} = -2 \pm \sqrt{1 - K'}$$



K'= 0: poles in -1 and -3

K' < 1: two real poles between -1 and -3

K'= 1: two real poles in -2

K'> 1: two complex poles with Re part -2

Higher-order

- Difficult to solve manually
- Use **20-sim**, MATLAB, …
- Use graphical method:
- Root locus
  - The root locus gives the locations of the poles of the closed system for variations in the loop gain of the system

Graphical evaluation (zero)

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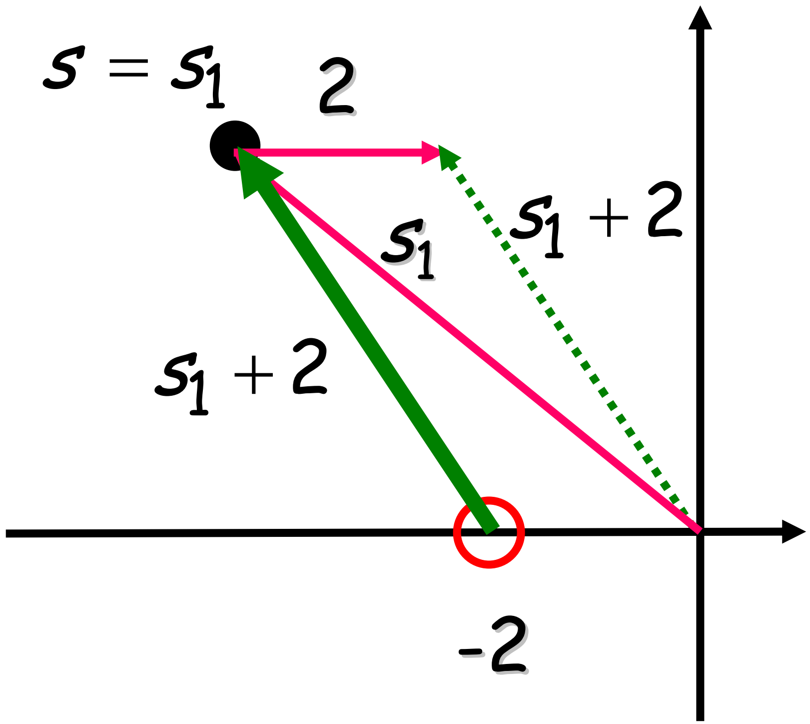
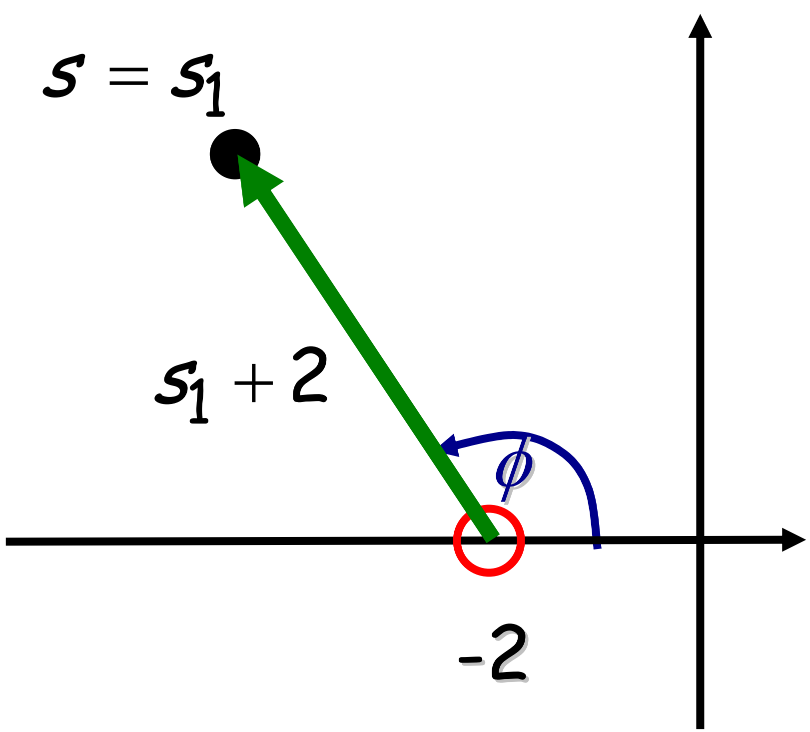


$$H(s) = s + 2$$

Graphical evaluation (zero)



$$H(s) = s + 2$$

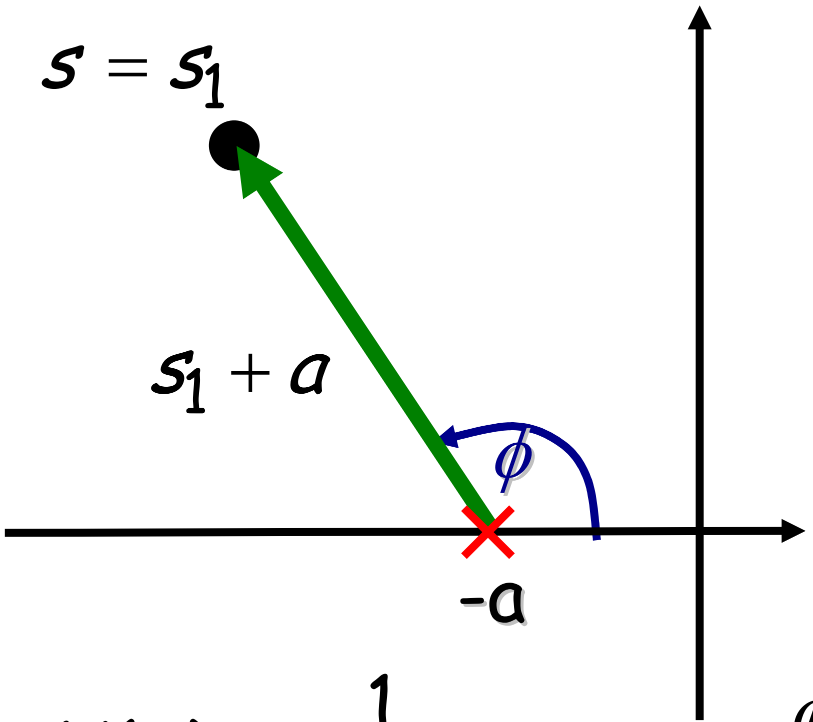
The value of $(s + a)$ in a point s_1 in the s-plane is equal to the vector from -a to s_1

$$s + a = |s + a| e^{j\phi}$$

$|s + a| =$ length of vector

$\phi =$ angle with pos. Re-axis

Graphical evaluation (pole)



$$H(s) = \frac{1}{s+a}$$

$$H(s)_{s=s_1} = \frac{1}{|s_1 + a|} e^{-j\phi}$$

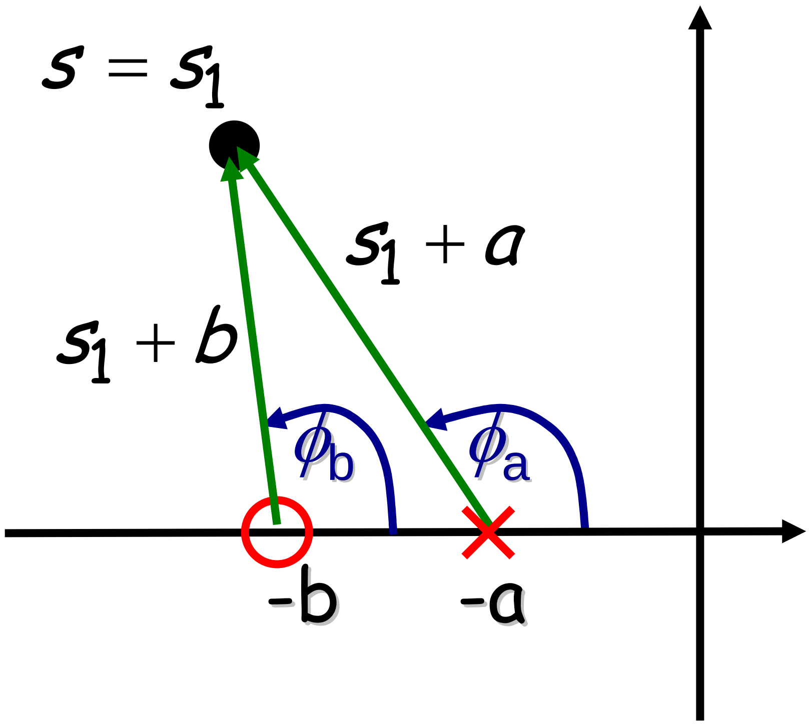
- sign

$$s + a = |s + a| e^{j\phi}$$

$$\frac{1}{|s+a|} = \frac{1}{\text{length of vector}}$$

$\phi =$ angle with pos. Re-axis

Graphical evaluation

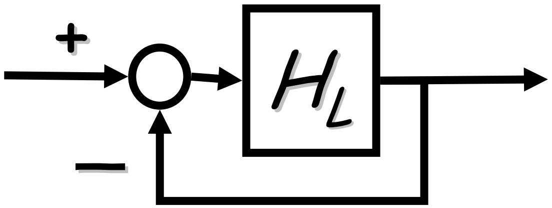


$$H(s) = \frac{s+b}{s+a}$$

$$H(s)_{s=s_1} =$$

$$= \frac{|s_1+b|}{|s_1+a|} e^{j(\phi_b - \phi_a)}$$

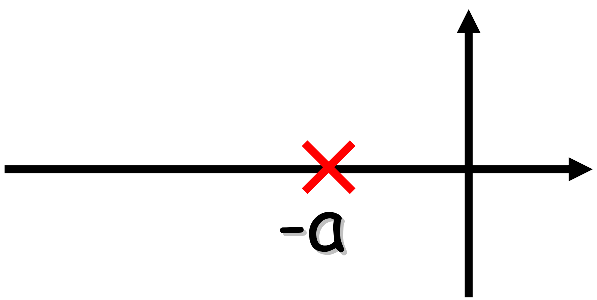
Root locus equation



$$H_L(s) = \frac{K'}{s+a}$$

$$H = \frac{H_L}{1+H_L}$$

$$H = \frac{K'}{s+a+K'}$$



poles of closed loop:

$$s + a + K' = 0$$

root locus equation: $$\frac{1}{s+a} = -\frac{1}{K'}$$

Root locus (first order)

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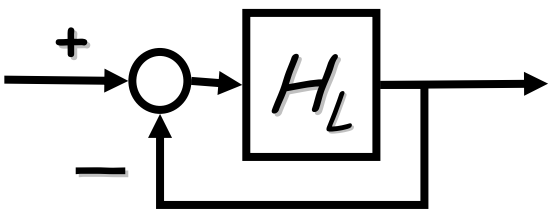
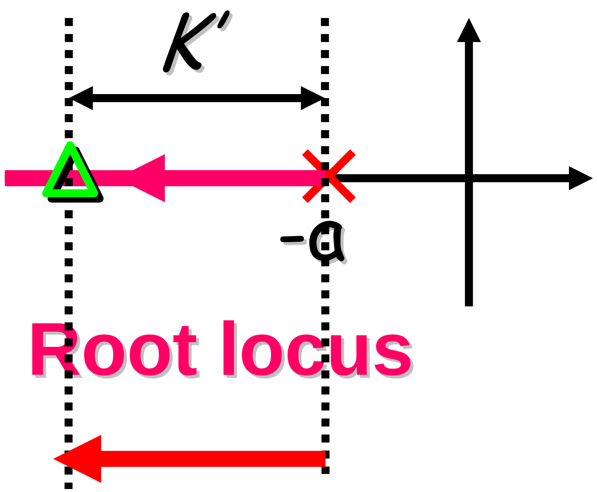


root locus equation: $\dfrac{1}{s+a} = -\dfrac{1}{K'}$

K = real



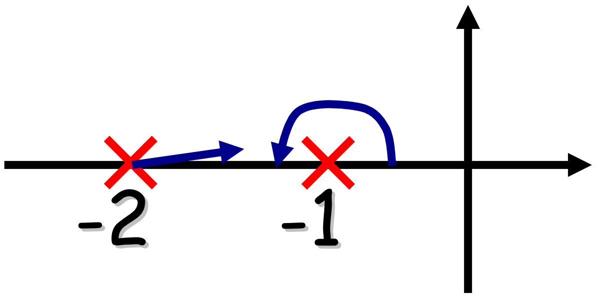
$$|s+a| = K'$$

$$\frac{1}{s+a} = \frac{1}{K'} e^{j\pi}$$

$$s + a = K' e^{j\pi}$$

points where $\angle_{s+a} = \pi \pm 2k\pi$,
are left of the pole in $-a$

Root locus (second order)

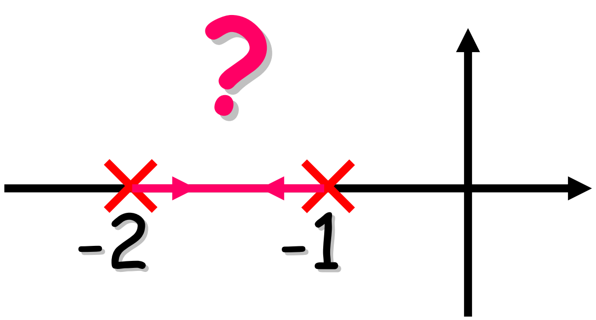


$$H_L = \frac{K_L'}{(s+1)(s+2)} = K_L' G_L'$$

root locus eq.

$$G_L' = -\frac{1}{K_L'} = \frac{1}{(s+1)(s+2)}$$

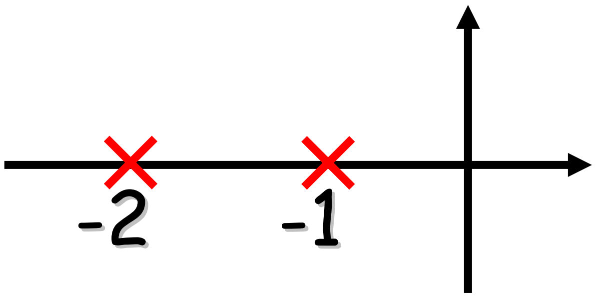
$$\rightarrow \underbrace{(s+1)(s+2)} = K_L' e^{j\pi}$$

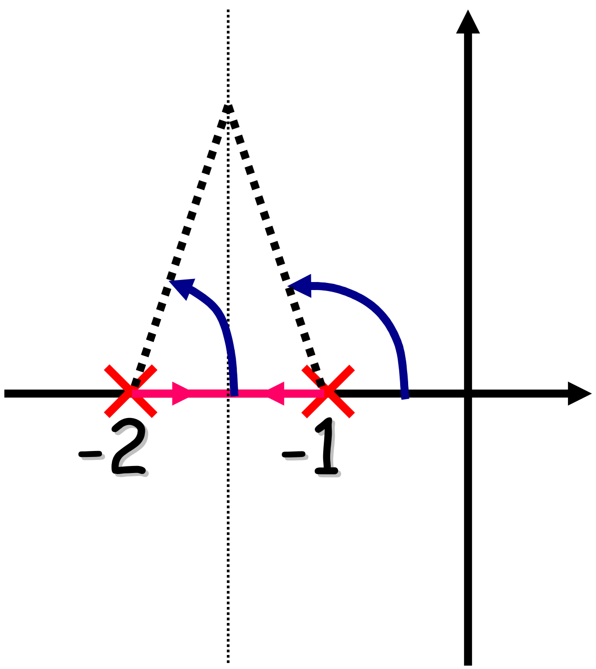


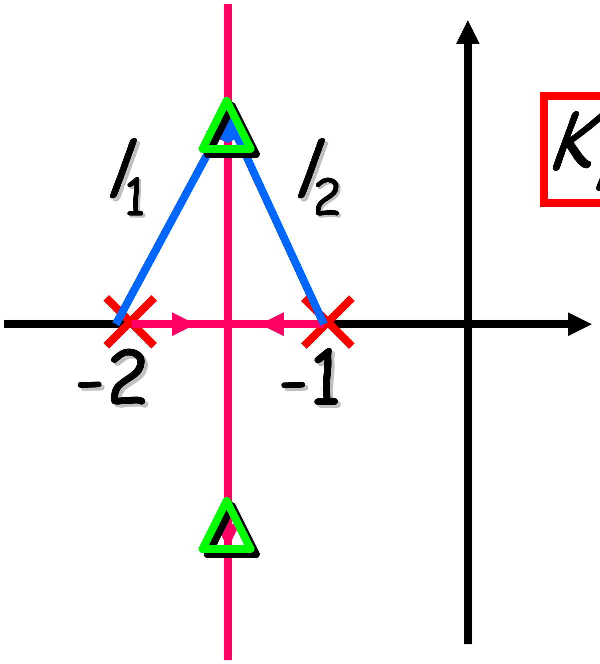
modulus = K_L'

argument = $e^{j(\pi+2k\pi)}$

Root locus (second order)







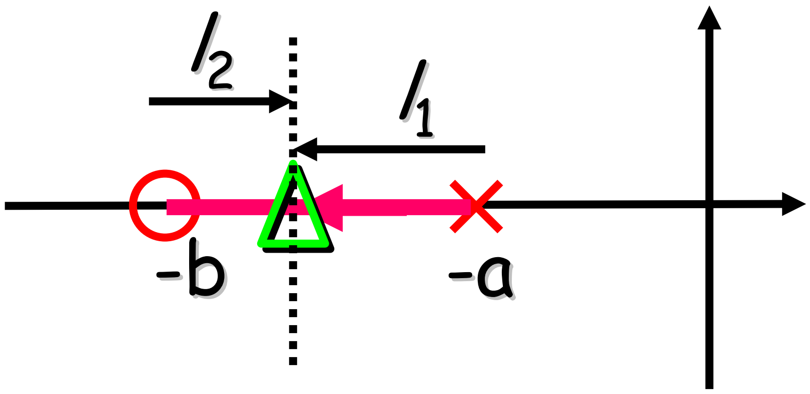
$$K_L' = l_1 \cdot l_2$$

$$(s+1)(s+2) = K_L' e^{j\pi}$$

$$\rightarrow K_L' = \left|(s+1)\right|\left|(s+2)\right|$$

Root locus (pole and zero)

$$H(s) = K_L' \frac{s+b}{s+a}$$



$$\frac{s+b}{s+a} = -\frac{1}{K_L'}$$

$$K_L' = \frac{|s+a|}{|s+b|}$$

$$K_L' = \frac{l_1}{l_2}$$

Demo's 20-sim

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- first order (pole in -1)
- second order (poles in -1 and -2)
- pole and zero

- step responses for various values of K'

Demo **20-sim**

Rules

- "argument = -180 degrees" rule too complex for manual construction of complex root loci

- Derive a set of rules that allows easy construction
- (based on the above rule)

Rule 1 (start and end)

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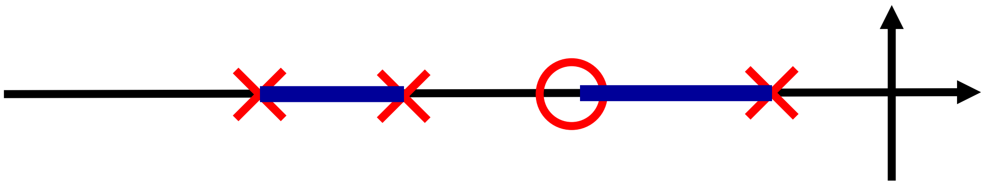
root locus equation: $G'_L = -\frac{1}{K'_L}$

- The root locus for variations in K'_L starts for $K'_L = 0$ in the poles of G'_L and ends for $K'_L \rightarrow \infty$ in the zero's of G'_L or in ∞ (if there are more poles than zero's)

Rule 2 (real axis)

root locus equation: $G'_L = -\frac{1}{K'_L}$

- The locus includes all points along the real axis to the left of an odd number of poles plus zero's of G'_L



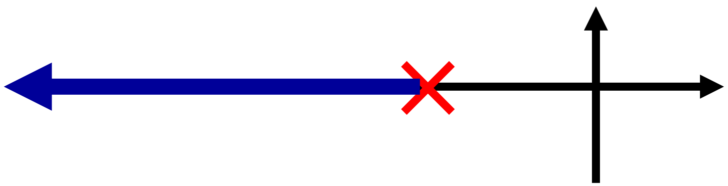
Rule 3 (asymptotes)

- for $K'_L \to \infty$ the branches of the locus become asymptotic to straight lines with angles

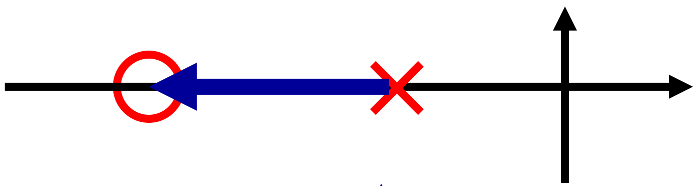
$$\theta = \frac{\pi \pm 2k\pi}{n-m}$$

for $k = 0, \pm 1, \pm 2, \ldots,$ until all n - m angles are obtained, where n is the the number of poles and m the number of zero's

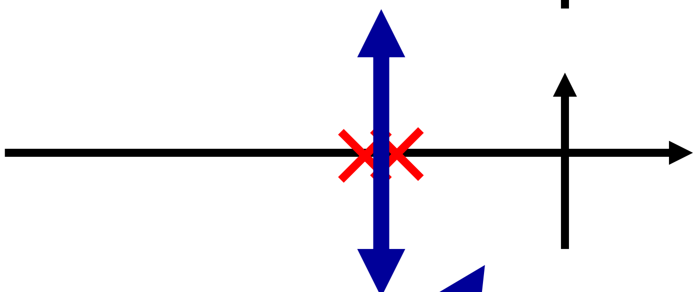
Rule 3 examples



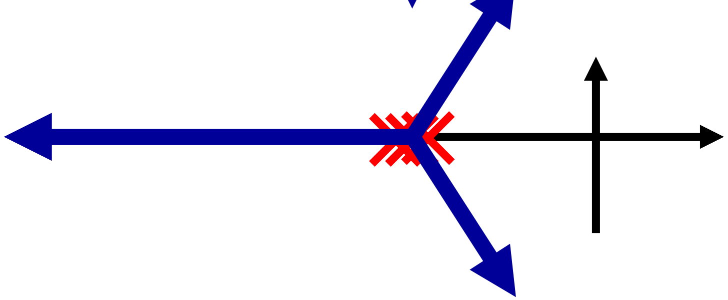
$n - m = 1$: 1 asymptote: $-\pi$



$n - m = 0$: no asymptote



$n - m = 2$: 2 asymptotes:
$\pm\pi/2$

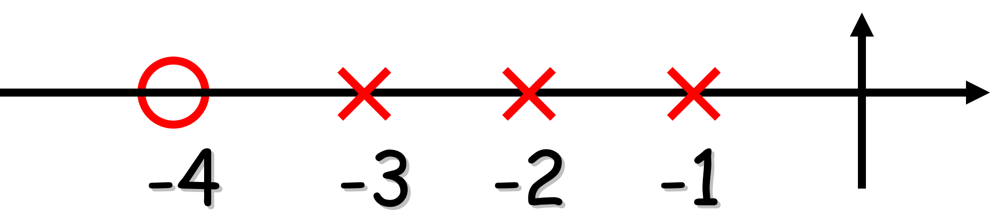


$n - m = 3$: 3 asymptotes:
$\pm\pi/3$

Rule 4 (start of asymptotes

- starting point of the asymptotes, the centroid of the pole-zero plot, is on the real axis at:

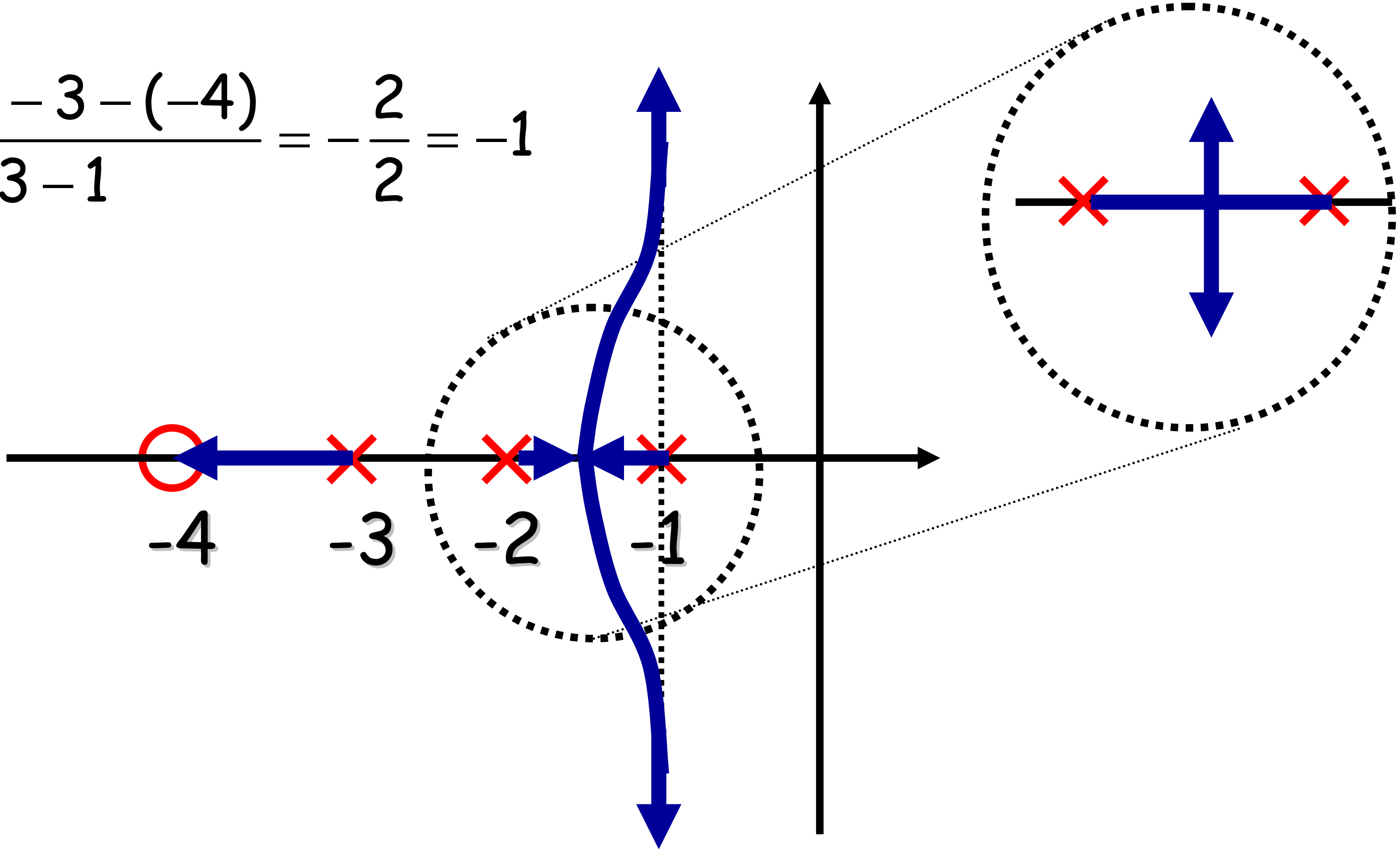
$$\sigma = \frac{\sum \text{pole values of } G_L' - \sum \text{zero values of } G_L'}{n - m}$$



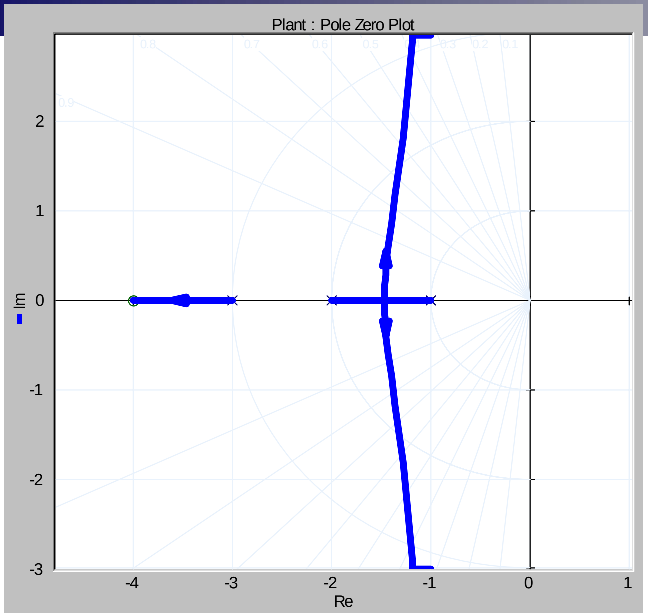
$$\sigma = \frac{-1-2-3-(-4)}{3-1} = -\frac{2}{2} = -1$$

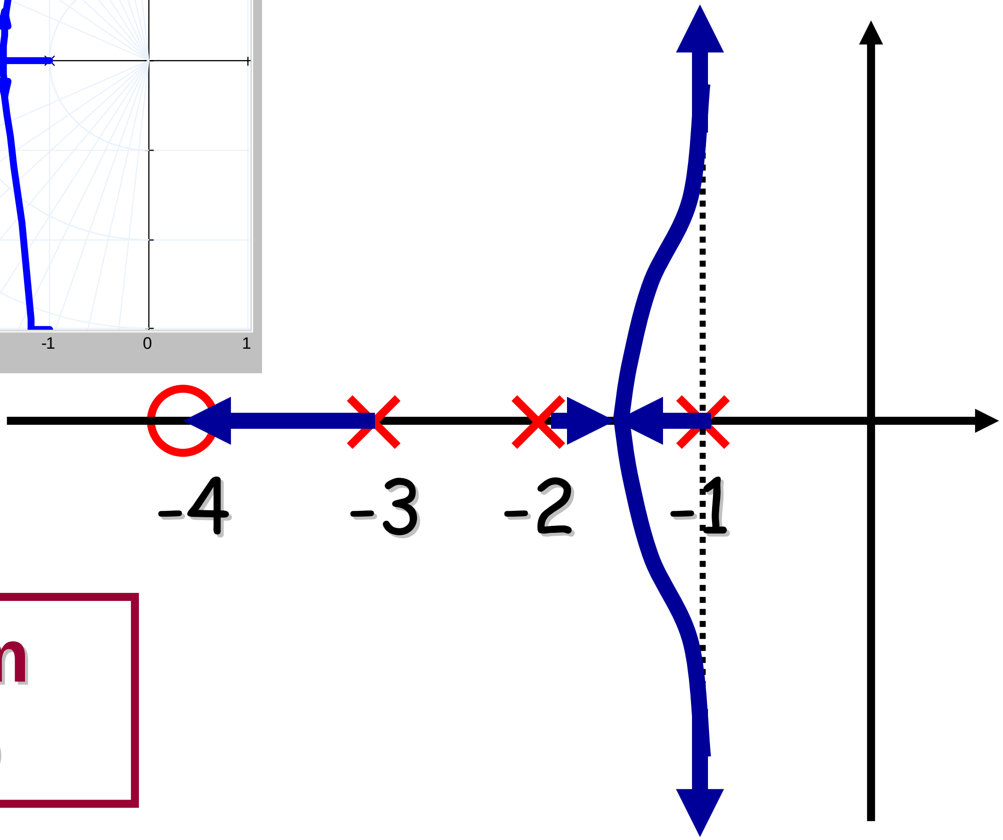
Rule 4 (Example)

$$\sigma = \frac{-1-2-3-(-4)}{3-1} = -\frac{2}{2} = -1$$



Rule 4 (20-sim demo)





20-sim
demo

Rule 5 (breakaway & entry)

- Loci leave (enter) the real axis at a gain K'_L that is the maximum (minimum) value of K'_L in that region on the real axis. These points are called breakaway (entry) points:

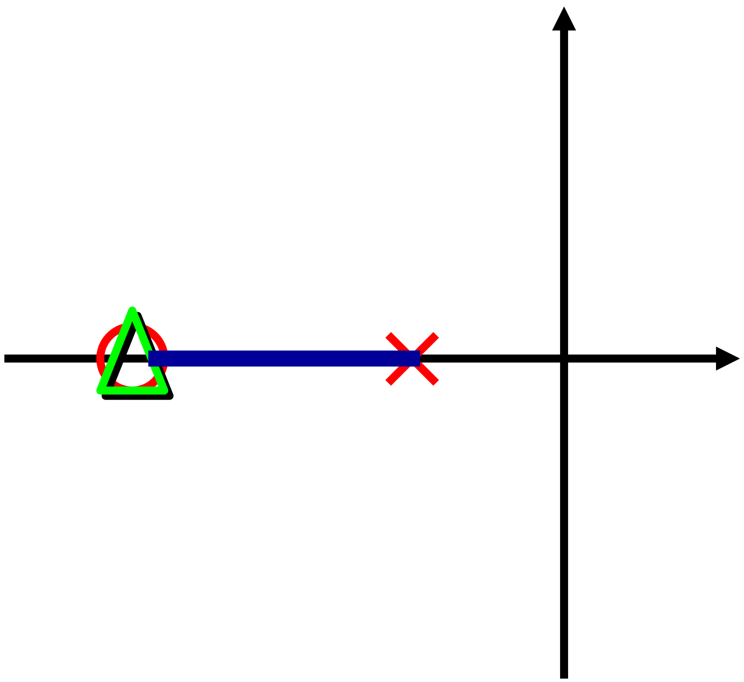
$$\frac{dK'_L}{ds} = -\frac{d}{ds}\left(\frac{1}{G'_L}\right) = 0$$

- Locus segments leave (enter) the real axis at angles of $\pm\pi/2$

Electrical charges model

model poles as bodies with a positive charge

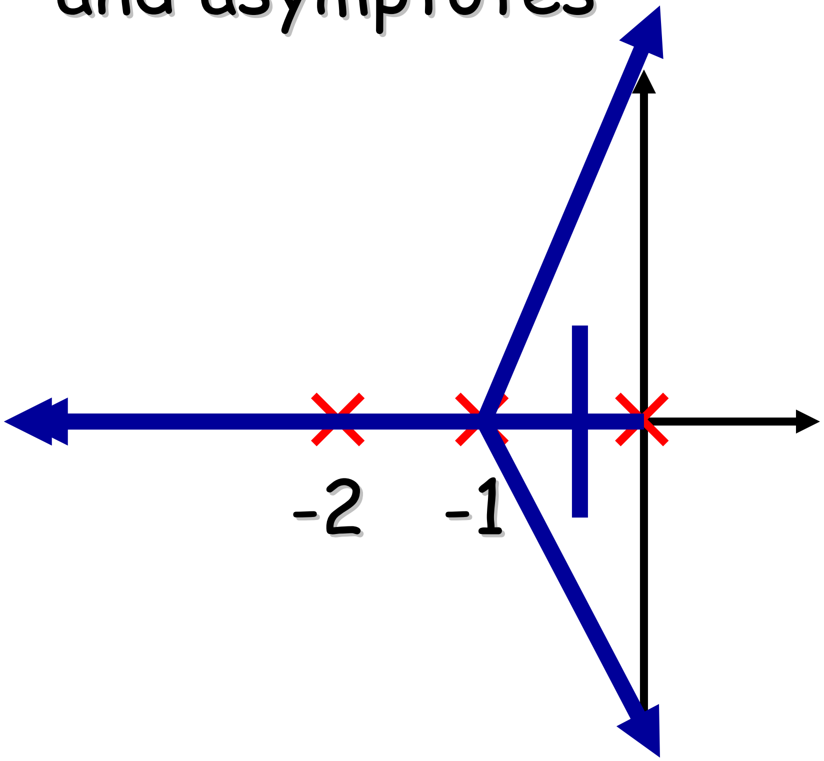
model zeros as bodies with a negative charge



a positive charge just left of the pole of the open system is repulsed by the pole and attracted by the zero

Electrical charges model

Also helps to find breakaway points and asymptotes



Far away the 3 poles behave as three poles in -1

Because of positive charge in -2, breakaway point will be at the right of -0.5

Rule 6 Angles of departure

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- angle of departure ϕ of a locus branch from a complex pole is given by:

$$\phi = -\sum \text{arg(other poles)} + \sum \text{arg(zeros)} + \pi$$

- The angle of approach ϕ of a locus branch to a complex zero is given by:

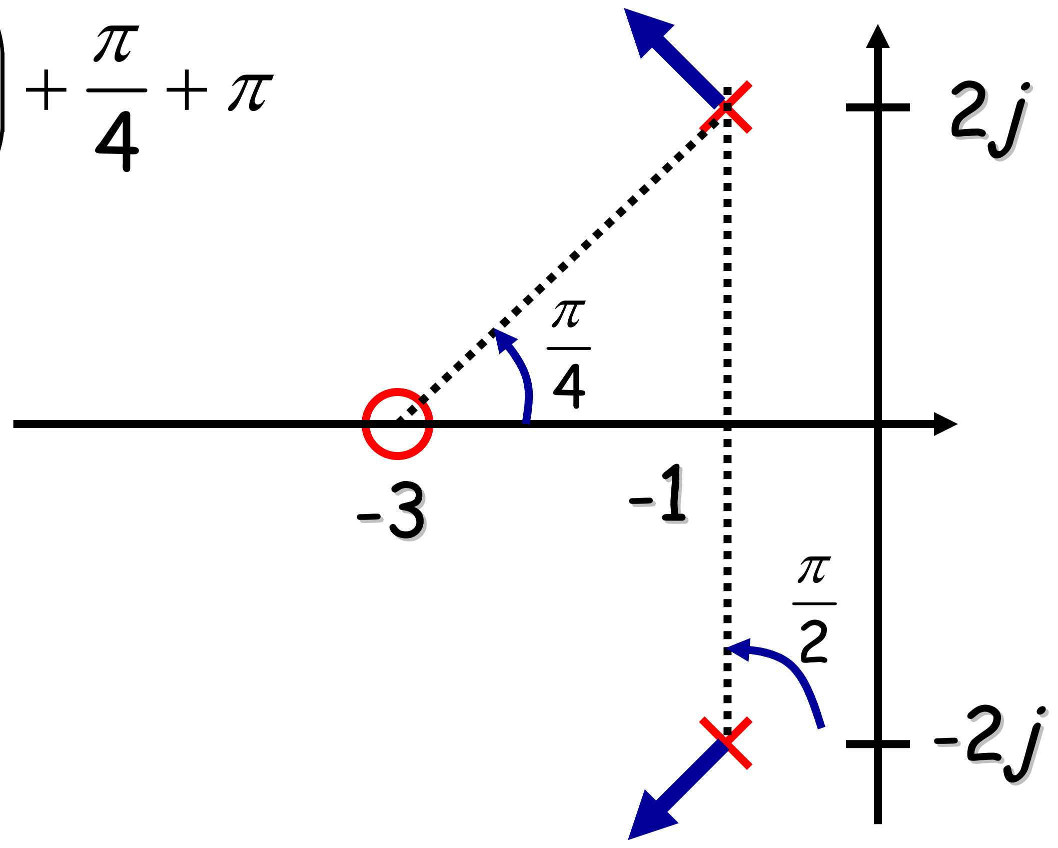
$$\phi' = \sum \text{arg(poles)} - \sum \text{arg(other zeros)} - \pi$$

Rule 6 (example)

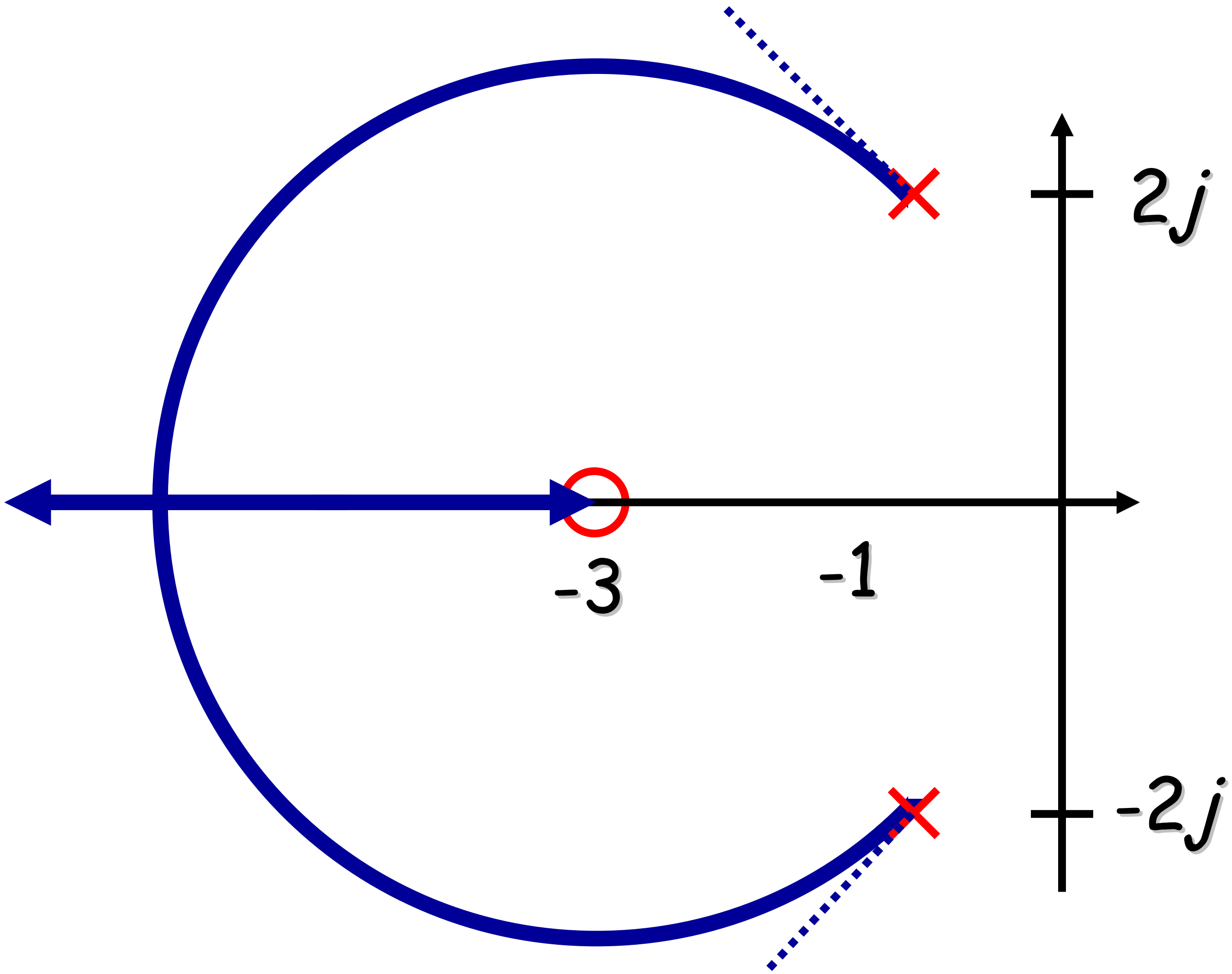
$$\phi = -\sum \text{arg(other poles)} + \sum \text{arg(zeros)} + \pi$$

$$\phi = -\left(\frac{\pi}{2}\right) + \frac{\pi}{4} + \pi$$

$$\phi = \frac{3\pi}{4}$$

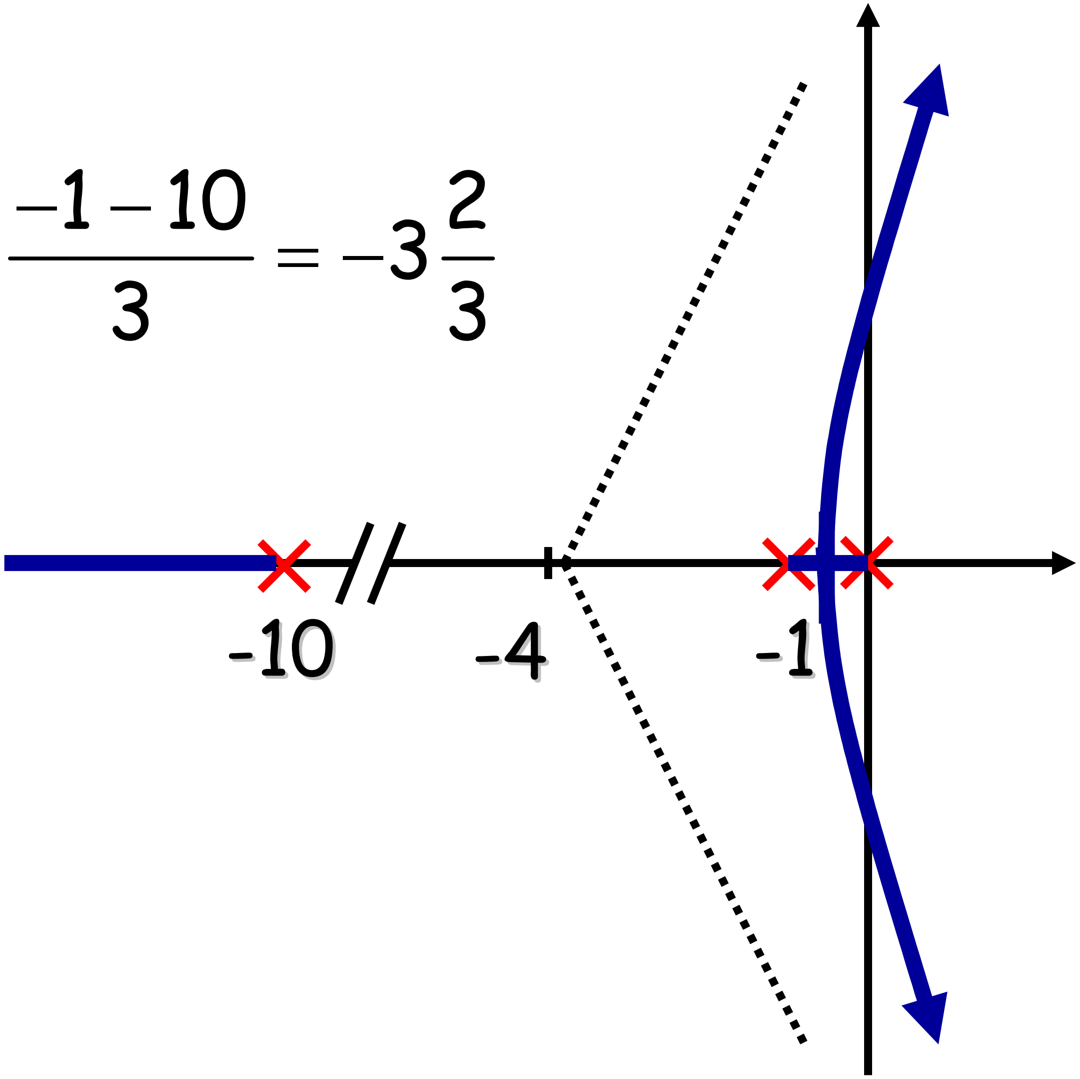


Rule 6 (example)



Example (Servo System)

$$\frac{-1-10}{3} = -3\frac{2}{3}$$



rule 1,2

rule 3,4

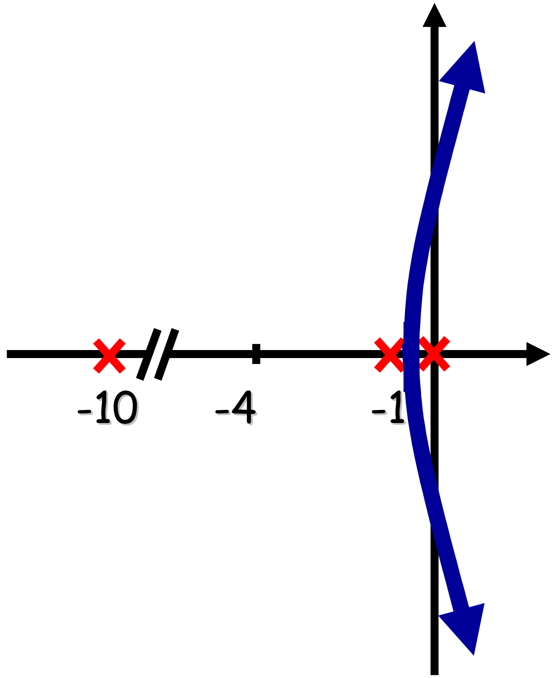
rule 5

Example

$$s(s+1)(s+10)+K_L'=0$$

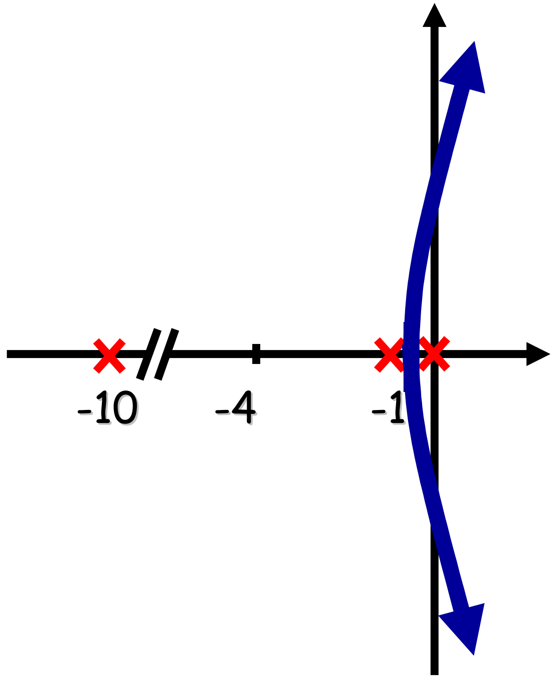
stability boundary

$$j\omega(j\omega+1)(j\omega+10)+K_L'=0$$



Example

$$s(s+1)(s+10)+K_L'=0$$



stability boundary

$$j\omega(j\omega+1)(j\omega+10)+K_L'=0$$

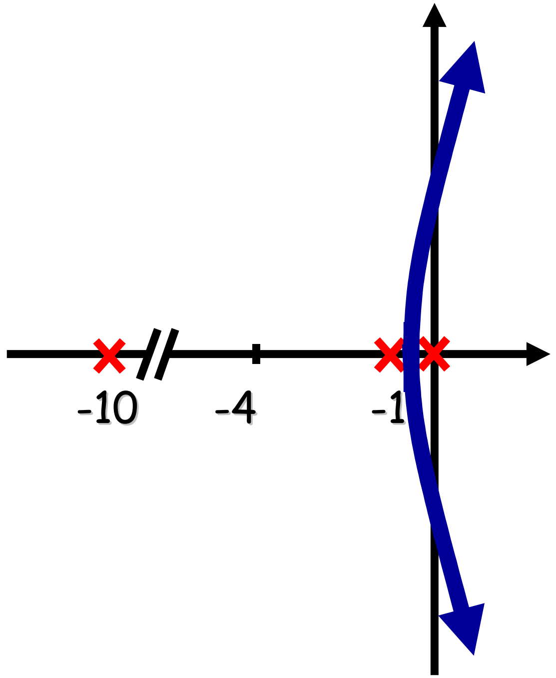
$$j\omega\left(-\omega^2+11j\omega+10\right)+K_L'=0$$

$$\left(-j\omega^3-11\omega^2+10j\omega\right)+K_L'=0$$

$$\left(-\omega^3+10\omega\right)=0$$

Example

$$s(s+1)(s+10)+K_L'=0$$



stability boundary

$$j\omega(j\omega+1)(j\omega+10)+K_L'=0$$

$$j\omega\left(-\omega^2+11j\omega+10\right)+K_L'=0$$

$$\left(-j\omega^3-11\omega^2+10j\omega\right)+K_L'=0$$

$$\left(-\omega^3+10\omega\right)=0$$

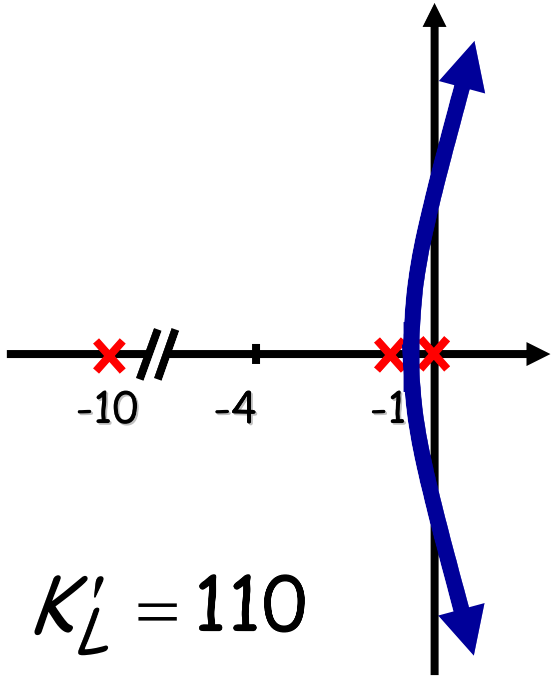
$$\omega=0,\ \omega=\sqrt{10}$$

$$110=K_L'$$

Example

$$s(s+1)(s+10)+K_L'=0$$

stability boundary



$$j\omega(j\omega+1)(j\omega+10)+K_L'=0$$

$$j\omega\left(-\omega^2+11j\omega+10\right)+K_L'=0$$

$$\left(-j\omega^3-11\omega^2+10j\omega\right)+K_L'=0$$

$$\left(-\omega^3+10\omega\right)=0$$

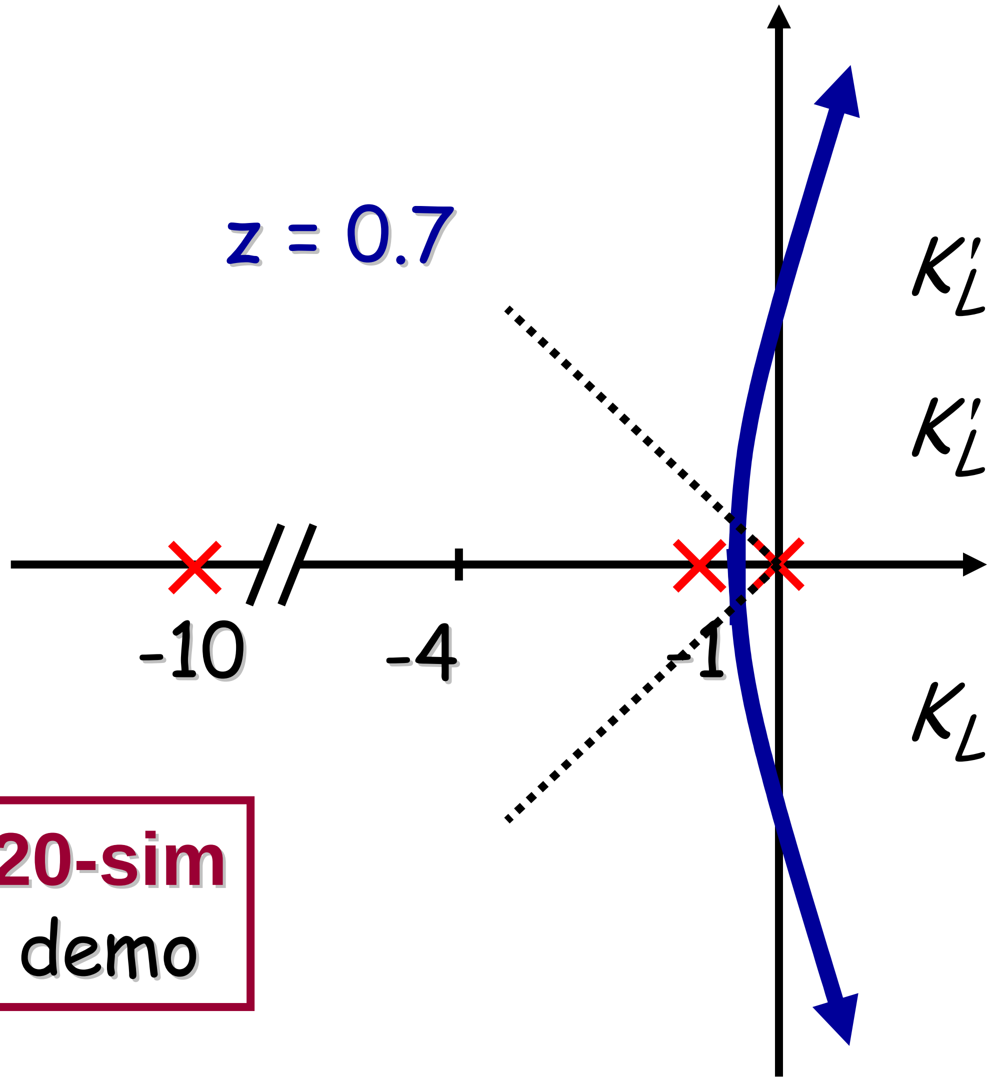
$$\omega=0,\ \omega=\sqrt{10}$$

$$110=K_L'$$

$$K_L'=110$$

$$K_L=\frac{110}{10\cdot 1}=11$$

Example



$$K_L' \approx 0.5\sqrt{2} \times 0.5\sqrt{2} \times 10$$

$$K_L' \approx 0.5 \times 10 = 5$$

$$K_L \approx \frac{5}{10 \cdot 1} = 0.5$$

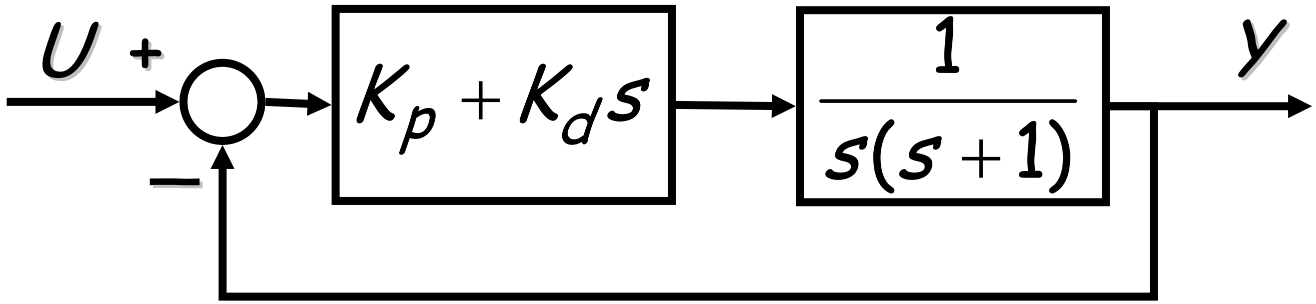
20-sim
demo

More than 1 controller parameter

Example:

- Proportional feedback (K_p)
- Velocity / tacho feedback (K_d)

τ-locus



root locus equation: $s(s+1) + K_p + K_d s = 0$

for variations in K_d:

zero (only for locus!)

$$\frac{1}{s(s+1)} = -\frac{1}{K_p} \qquad \frac{s}{s(s+1)+K_p} = -\frac{1}{K_d}$$

τ-locus (steps)

Use RLE for $K_d = 0$
to find poles for desired K_p
e.g. $K_p = 10$

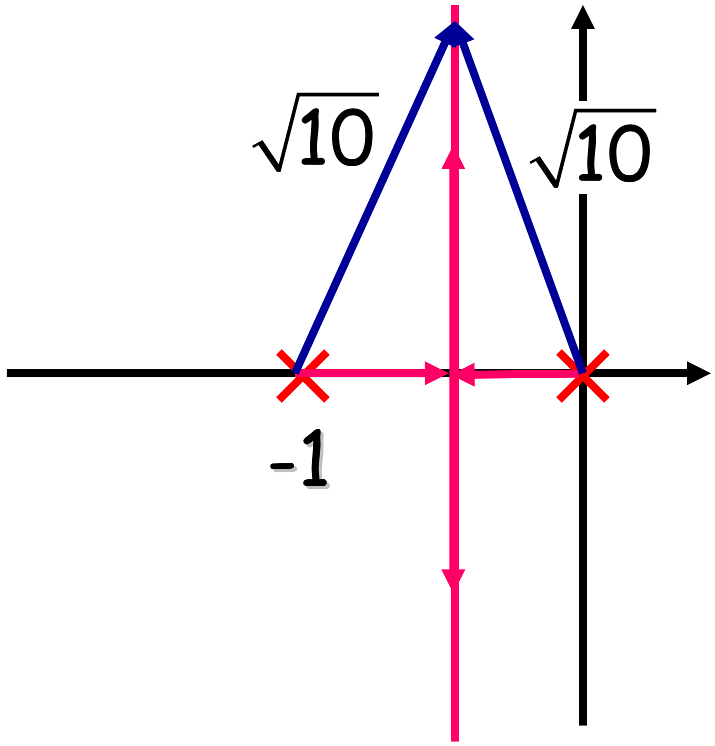
$$\frac{1}{s(s+1)} = -\frac{1}{K_p}$$

Use RLE for K_d
to find poles with $K_p = 10$

$$\frac{s}{s(s+1)+10} = -\frac{1}{K_d}$$

$$\frac{s}{(s+p_1)(s+p_2)} = -\frac{1}{K_d}$$

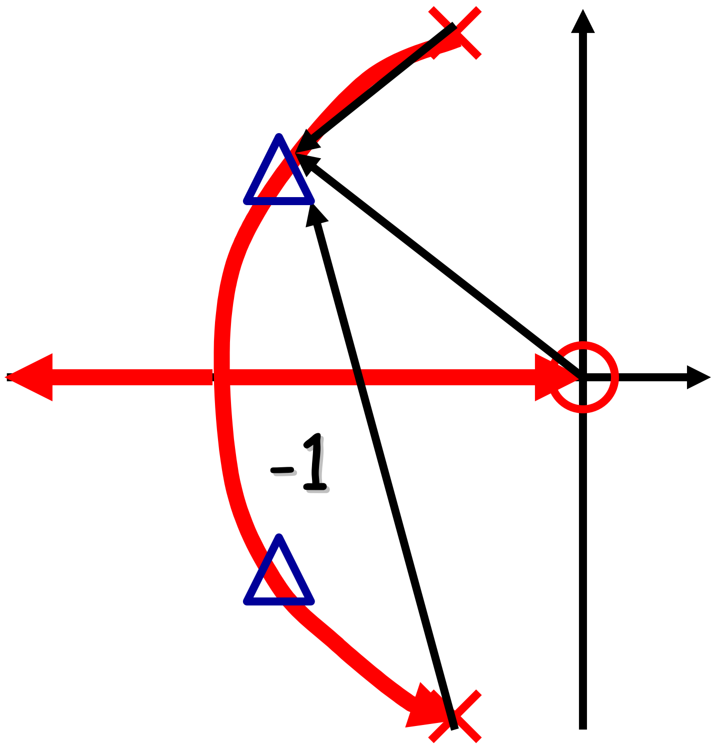
Loci



$$\frac{1}{s(s+1)} = -\frac{1}{K_p}$$

$$K_p = l_1 \cdot l_2 \rightarrow l_1, l_2 = \sqrt{10}$$

Loci

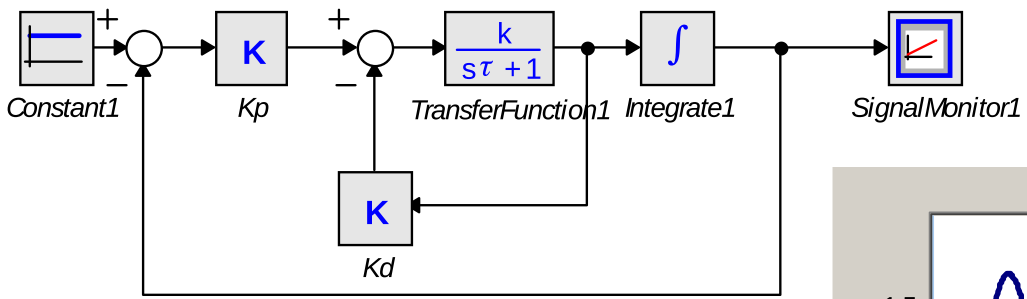


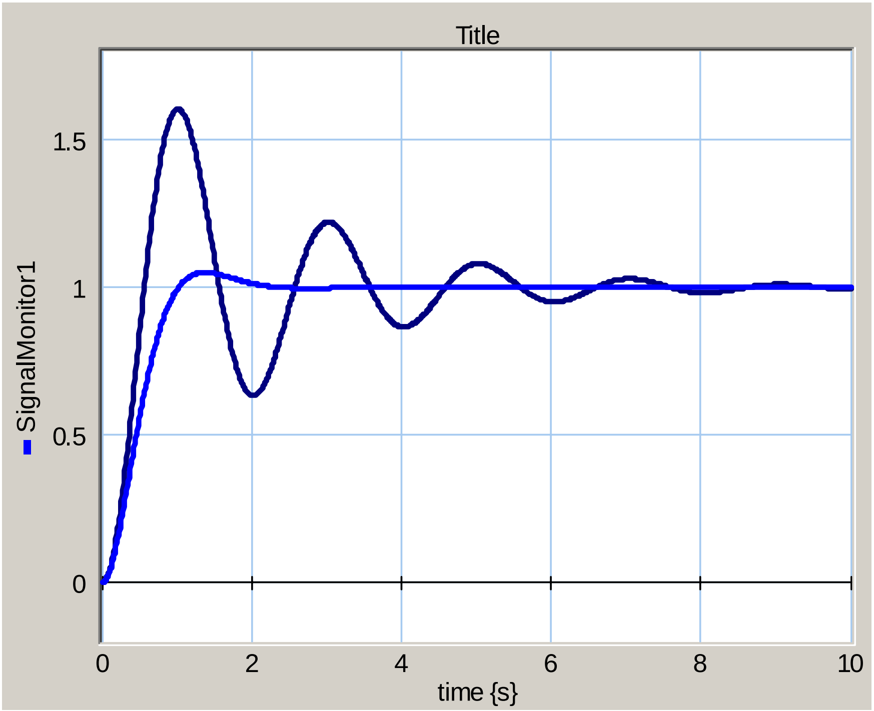
$$\frac{s}{s(s+1)+10} = -\frac{1}{K_d}$$

Take care that root locus gain of "proces" = 1

20-sim
demo

Controlled System





20-sim
demo